

Noncommutative geometry and quantum field theory

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The beginning of noncommutative geometry is the observation that there is a rough equivalence — contravariant — between the category of topological spaces on the one hand, and the category of commutative algebras over \mathbb{C} on the other. In one direction, we associate to a space X the algebra $C(X)$ of continuous functions $X \rightarrow \mathbb{C}$. In the other direction we associate to a commutative algebra A the space $\text{Spec}(A)$ of algebra homomorphisms $A \rightarrow \mathbb{C}$, or, equivalently, of irreducible A -modules.

If we agree that commutative algebras can usefully be thought of as corresponding to spaces in this way, then noncommutative geometry is the attempt to develop an analogous theory for noncommutative algebras. Why might one want to do such a thing? Three possible motivations are worth mentioning.

1. Even if one's interest in a commutative ring is purely algebraic, it is unquestionable that its properties are greatly illuminated by thinking in terms of the space it defines. One can aim for similar illumination about noncommutative rings. This is not, however, my main concern to-day.
2. We often encounter mathematical objects which we feel intuitively are "spaces", but whose space-like properties seem to melt away on closer examination. A standard example is the space of leaves of a foliated manifold when each leaf is dense in the manifold; another — very similar — is the space of orbits of a group which acts ergodically on a space. These are the spaces which Connes uses to motivate the study of noncommutative geometry, for, although there are no non-constant continuous complex-valued functions on them, nevertheless their points are the irreducible representations of a noncommutative algebra naturally associated to the foliation or group action.

3. The third motivation comes from quantum theory, according to which the world is properly described by something like a noncommutative algebra, to which the state-spaces of classical physics are simply convenient commutative approximations. I shall return to this below.

My talk will have four parts. [BUT ONLY THE FIRST TWO PARTS ARE INCLUDED IN THIS MANUSCRIPT]

The first explains how a noncommutative space is really defined by an additive category rather than by an algebra.

The second describes the kind of homotopy type that a noncommutative space possesses.

The third describes how such homotopy types arise in ordinary geometry, in Floer's infinite-dimensional Morse theory.

The fourth explains how string theory leads to a slightly different notion of a noncommutative space, in which additive categories are replaced by the A_∞ categories of Kontsevich. These structures are in some ways more general but also in important ways *less* general than traditional noncommutative spaces.

Additive categories

In Connes's examples, the underlying set is the set of isomorphism classes of a *category* — normally of a *topological* category [S1], i.e. one where there is a topology on the set of objects and also on the set of morphisms. (In the examples, the category is in fact a *groupoid*, i.e. every morphism has an inverse.)

To any category \mathcal{C} — which here, for simplicity, I shall assume to be discrete — one can associate an algebra $A_{\mathcal{C}}$ over the complex numbers in such a way that functors from \mathcal{C} to complex vector spaces correspond precisely to $A_{\mathcal{C}}$ -modules, and isomorphism classes of objects of \mathcal{C} correspond precisely to irreducible $A_{\mathcal{C}}$ -modules. In fact $A_{\mathcal{C}}$ has a vector space basis $\{e_f\}$ indexed by the morphisms f of \mathcal{C} , and its multiplication is defined by

$$e_f e_g = e_{f \circ g}$$

when the composition $f \circ g$ is defined, and $e_f e_g = 0$ otherwise.

Example If \mathcal{C} has n objects, and a unique isomorphism between any pair, then $A_{\mathcal{C}}$ has a basis $\{e_{ij}\}$ with $1 \leq i, j \leq n$, and $A_{\mathcal{C}}$ is the algebra $\text{Mat}_n(\mathbb{C})$ of $n \times n$ matrices.

As we are interested in the isomorphism classes of objects of the category \mathcal{C} , the category of $A_{\mathcal{C}}$ -modules is more relevant than the algebra $A_{\mathcal{C}}$ itself. In fact equivalence of categories corresponds to *Morita equivalence* of the corresponding algebras: recall that two rings A and B are said to be Morita equivalent if their categories of modules are equivalent. It is more useful to think of the whole category of modules rather than just the set of irreducible modules, because an algebra can easily have no irreducible modules at all. (One of Connes's favorite examples is the space of Penrose tilings of the plane \mathbb{R}^2 , which corresponds to an algebra A for which the isomorphism classes of finitely generated modules are indexed by the positive real numbers of the form $n + m\omega$, where n and m are integers and ω is the golden ratio. There is no smallest such number, and hence no irreducible module.)

One reason for paying so much attention to the spaces of isomorphism classes of the objects of categories is that the fundamental state-spaces of twentieth-century physics are all of this form. In the language of physics this is expressed by saying that all fundamental interactions in nature are described by “gauge theories”. For example, in the 19th century one thought of the electromagnetic field as a closed 2-form on space-time; but nowadays one knows that it is more correctly described as a complex line-bundle on space-time equipped with a connection — the curvature of the connection being the classical field strength, which determines the bundle and its connection up to isomorphism if the space-time is simply connected, but not otherwise.

Similarly, the state-space of general relativity is the space of isomorphism classes of pseudo-Riemannian 4-manifolds whose metrics obey Einstein's equations: it is *not* the space of such metrics on a given 4-manifold.

Homotopy types

We pass from the category of topological spaces and continuous maps to the *homotopy category* by identifying maps which are homotopic. The homotopy type of a space is very much more robust than the space itself. To understand this it is helpful to recall [S1] that any category — or topological

category — \mathcal{C} has a space $|\mathcal{C}|$ associated to it called its *realization*, and that the homotopy type of $|\mathcal{C}|$ depends on \mathcal{C} only up to equivalence of categories. It turns out that there are many categories which can be associated to a well-behaved topological space X whose realizations have the same homotopy type as X . For example, one can take the category of all contractible open subsets of X and their inclusions; but also one can take the category of all sufficiently small open subsets, or the category of covering spaces of such subsets, or the category of all singular simplexes in X (the morphisms being inclusions of faces). By managing to define a suitable category of generalized open subsets for any commutative ring A , Grothendieck succeeded in associating a homotopy type to A which coincides with that of $\text{Spec}(A)$ for a class of algebras A over the complex numbers, but which is much more interesting than the badly-behaved space $\text{Spec}(A)$ in general.

Each of Connes’s basic examples comes from a topological category \mathcal{C} , and so the homotopy type of $|\mathcal{C}|$ is naturally associated to the noncommutative space. Can we associate such a homotopy type to an arbitrary noncommutative algebra? The answer is no.

A clue to understanding this is that if the homotopy type does not change when we replace the algebra A by the Morita-equivalent algebra $\text{Mat}_2(A)$ of 2×2 matrices over A then homotopy classes of “noncommutative maps” can be *added*: when we have algebra-homomorphisms $f, g : B \rightarrow A$ we can define the sum $f \oplus g : B \rightarrow \text{Mat}_2(A)$.

There are at least three well-known ways of mapping the homotopy category to an additive category. In each case the objects are still topological spaces, but there are three different abelian groups of morphisms from a space X to a space Y , which can be regarded as successively cruder “additive envelopes” of the set $[X; Y]$ of maps in the homotopy category. I shall denote them

$$\{X; Y\} \rightarrow k(X; Y) \rightarrow H(X; Y).$$

The simplest to describe is $H(X; Y)$, which is the group $[X; \mathbb{Z}_Y]$ of homotopy classes of “many-valued” maps from X to Y , a many-valued map being defined as a continuous map from X to the free abelian group \mathbb{Z}_Y on the space Y (regarded as a topological abelian group with the topology it inherits from Y). At the other end is the algebraic topologists’ space of *stable* homotopy classes $\{X; Y\} = [X; Q_Y]$ of maps, which is the *universal* repre-

sentable abelian-group-valued functor¹ enveloping $[X; Y]$. In this talk I shall not discuss the stable homotopy category beyond pointing out that its construction involves the “thickening” of spaces: when X and Y are compact, at least, the elements of $\{X; Y\}$ are continuous maps from $X \times \mathbb{R}^N$ to the one-point compactification of $Y \times \mathbb{R}^N$, for large N , which tend to infinity at infinity.

The important group for the moment is $k(X; Y) = [X; k_Y]$, for it can be defined directly in terms of the algebras $C(X)$ and $C(Y)$ of continuous complex-valued functions on X and Y , in a way that makes good sense for arbitrary noncommutative algebras. The space k_Y should be called the *noncommutative spectrum* of the algebra $C(Y)$: if Y is compact it is formed by adjoining formal inverses to the topological semigroup

$$k_Y^+ = \coprod_{N \geq 0} \text{Hom}_{alg}(C(Y); \text{Mat}_N(\mathbb{C}))$$

of algebra-homomorphisms from $C(Y)$ to the algebras of $N \times N$ matrices, for all N . (The composition-law in k_Y is the direct sum operation already mentioned. For more details see [S2].) Up to homotopy, we can say that k_Y is the space of virtual $C(Y)$ -modules of finite length. It is a kind of “categorification” of the abelian group \mathbb{Z}_Y , for a point of \mathbb{Z}_Y is a finite formal combination $\sum n_i y_i$ of points y_i of Y labelled with integers n_i , while a point of k_Y is a similar combination of points labelled by virtual finite dimensional vector spaces.

Remarks

(a) Despite their very different definitions, the three additive categories made from the homotopy category are more similar than they look, for the groups $\{X; Y\}$ and $H(X; Y)$ become isomorphic after tensoring with the rational numbers \mathbb{Q} : each is the vector space of homomorphisms of graded vector spaces from $H_*(X; \mathbb{Q})$ to $H_*(Y; \mathbb{Q})$. The group $k(X; Y) \otimes \mathbb{Q}$ differs only by partial stabilization with respect to suspension: it is the vector space of homomorphisms from $H_*(X; \mathbb{Q})$ to $H_*(Y; \mathbb{Q})$ of any even negative degree.

(b) In terms of ordinary integral homology, we have

$$H(X; Y) \cong \prod_{k \geq 0} H^k(X; H_k(Y)).$$

¹This means that Q_Y is an abelian group in the homotopy category, and the map $Y \rightarrow Q_Y$ is universal among maps in the category to such objects.

(c) The H -category is purely algebraic: it is the *derived category* of the ring \mathbb{Z} , or, equivalently, the category of chain complexes of free abelian groups and chain-homotopy classes of chain maps.

Could we expect to do better than this, and assign an honest homotopy type to a noncommutative ring? Very roughly speaking, the homotopy category is distinguished from the stable homotopy category by the fact that an actual space X has a diagonal map $X \rightarrow X \times X$ — essentially, the homotopy category is the category of commutative coalgebras in the stable category. For a commutative algebra A the diagonal map is the multiplication $A \otimes A \rightarrow A$, and we have no such homomorphism when A is not commutative.

But there is an important source of explicit examples of noncommutative spaces which do not have ordinary homotopy types. If X is an ordinary space, then a bundle \mathcal{B} of full matrix algebras — perhaps infinite dimensional — on X is called by physicists a *B-field* on X . Its algebra $A_{\mathcal{B}}$ of continuous cross-sections defines a noncommutative space $X_{\mathcal{B}}$. Because full matrix algebras are geometrically indistinguishable from points, we might at first expect $X_{\mathcal{B}}$ to be indistinguishable from X . But the bundle \mathcal{B} has an invariant $\beta_{\mathcal{B}}$ in $H^3(X; \mathbb{Z})$, called its *Dixmier-Douady invariant*, which classifies the algebras $A_{\mathcal{B}}$ up to Morita equivalence; and all elements of $H^3(X; \mathbb{Z})$ can arise. As soon as $\beta_{\mathcal{B}} \neq 0$ the “space” $X_{\mathcal{B}}$ has no diagonal map, and no conventional homotopy type.

Quantum theory

My own interest in noncommutative geometry is motivated by quantum theory. Whereas a classical physical system has at each time a “state” x which is a point in a manifold X of “possible states”, a quantum system is described at each time by a pair (A, θ) , where A is a $*$ -algebra and $\theta : A \rightarrow \mathbb{C}$ is a positive linear form. From the quantum description one obtains an approximate classical description by recognizing (A, θ) as a small deformation of another pair (A_0, θ_0) such that A_0 is commutative and θ_0 is a homomorphism of algebras. Then A_0 defines a classical state-space, and θ_0 defines a point $x \in X$. (More realistically, one replaces the algebra of observables A by a much smaller one which contains the observables one is interested in, and which *does* have a classical approximation.)

For our present purposes, the crucial thing about the quantum account is that it suggests a rather different way of associating a space to an algebra A :

instead of the set of irreducible representations of A we think of the points of the space associated to a commutative algebra of which A is a small deformation. On the level of points the two procedures are plainly quite different. They are different even in the homotopy category, but there the difference is more tractable, and can be obliterated by passing to a coarser version of the homotopy category. Even without the motivation from quantum theory, indeed, it would be natural to look for a homotopy type which is invariant under small deformations of the algebra — that is what one expects of anything called a homotopy type. It turns out that the departure from deformation invariance is completely encapsulated by a single example, which in turn is a formulation of the Stone-von Neumann theorem of elementary quantum mechanics.

The original version of the theorem states (roughly) that the $*$ -algebra generated by two self-adjoint operators P and Q which satisfy the commutation relation $PQ - QP = ih$ (for some non-zero real number h) has a *unique* irreducible $*$ -representation on Hilbert space. A more precise version, however, is as follows.

Let A_0 denote the algebra $C_0^\infty(\mathbb{R}^2)$ of Schwartz functions — smooth functions which decay rapidly at infinity — on the plane \mathbb{R}^2 , under point-wise multiplication. By Fourier transformation we can identify A_0 with the convolution-algebra of Schwartz functions on the additive group \mathbb{R}^2 . Now let us deform A_0 to another algebra A_h by adding a cocycle to the convolution product, i.e. we define

$$(f *_h g)(\xi) = \int_{\mathbb{R}^2} e^{ihS(\eta, \xi-\eta)} f(\eta)g(\xi - \eta)d\eta,$$

where $S(\eta, \zeta)$ denotes the area of the parallelogram spanned by η and ζ . Then the Stone-von Neumann theorem states that if $h \neq 0$ the algebra A_h is isomorphic to the algebra with the same underlying vector space, but where the functions are composed as the kernels of integral operators acting on the Hilbert space $L^2(\mathbb{R})$, i.e. according to the rule \circ , where

$$(f \circ g)(x, y) = \int_{\mathbb{R}} f(x, z)g(z, y)dz.$$

The significance of this theorem for homotopy theory is that A_0 is the algebra associated to the one-point compactification of the plane \mathbb{R}^2 , while when $h \neq 0$ the algebra A_h is essentially a full matrix algebra, and corresponds in

noncommutative geometry to a point. So if there is a deformation-invariant notion of homotopy type in noncommutative geometry it does not distinguish between a point and the compactification of \mathbb{R}^2 , and hence identifies the double suspension of every compact space X — i.e. the compactification of $X \times \mathbb{R}^2$ — with X itself. (As Connes has emphasized in his book [C], this is a version of the *Bott periodicity* theorem of homotopy theory.)

We can define a new additive category with compact spaces as its objects by taking the morphisms from X to Y to be

$$K(X, Y) = \lim_{n \rightarrow \infty} k((X \times \mathbb{R}^{2n})^+; Y),$$

where Z^+ denotes the one-point compactification of a compact space Z . This is the coarsening of the homotopy category in which a general algebra — more precisely, I should say a general C^* -algebra — has a deformation-invariant homotopy type. (In the usual language of homotopy theory, it is called the category of “**BU**-module spectra”.) Up to equivalence of categories one can take the objects of the category to be C^* -algebras, and in that version the category is called the *Kasparov category*.

As before, we have

$$K(X; Y) = [X; K_Y],$$

where K_Y is a space which in string theory is called the *space of D-branes* in Y . When Y is a smooth manifold its elements are represented by finite formal sums of even-dimensional submanifolds of Y equipped with complex vector bundles.